An Analytical Model for Velocity Distribution in Transition Zone for Channel Flows over Inflexible Submerged Vegetation

J. M. V. Samani\(^1\)*, and M. Mazaheri\(^1\)

**ABSTRACT**

The estimation of velocity distribution plays a major role in the hydrodynamics of vegetated streams or rivers of extensive natural floodplains. The velocity profile in vegetated channels can be divided into three zones: uniform zone which is close to bed with uniform velocity distribution, logarithmic zone which involves the main channel with no vegetive cover and the transition zone that is affected by the upper zone flow. In order to arrive at an analytical solution to the force balance that governs the flow specific turbulence, characteristics of the flow through the vegetation are required. A new analytical model for the velocity distribution in the transition zone of vegetated (inflexible submerged vegetation) channels is hereby developed. The model is based on a force equilibrium equation and on Prandtl Mixing Length concept. Vegetation is treated as a homogeneous field of identical cylindrical stems and the flow field considered as uniform and steady. The proposed procedure is straightforward; it follows principles of fluid mechanics and shows good agreement with laboratory flume experiments. The new model can be employed for an exact estimation of discharge through naturally vegetated rivers. The model has been calibrated and verified. The results imply a desirable correlation between calculated and observed data.

**Keywords:** Analytical model, Prandtl Mixing Length, Transition zone, Vegetative channels, Velocity distribution.

**INTRODUCTION**

A primary question in hydraulic considerations is that of quantifying velocity distributions in open channels under various conditions. Vertical velocity distributions, which constitute the bases for this research, were extensively studied in open channels. However, under natural conditions, solutions to these problems often require information about the influence of vegetation on the forces acting upon the water element and, consequently, on the performance of velocity distribution, particularly above bankful stage. The presence of vegetation in floodplains may have significant influence on the overall discharge capacity of a river (Darby, 1999). In particular if floodplains are relatively wide as compared to the main channel, realistic predictions of stage-discharge relationship rely strongly on an accurate knowledge of velocity distribution. Therefore, it is crucial to understand the processes that contribute to velocity distributions, and the hydraulic impacts these processes may exert. Such situations occur not only during extreme events but also on a regular basis (for example in winter or spring) when many rivers inundate adjacent areas. Under natural conditions, flood plains are usually vegetated by grass, hedges and/or trees. Unfortunately, despite

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the importance of the problem, little information does exist.

Trying to represent detailed velocity distribution in the vegetated channels is an immense task. Previous investigations have mostly focused on the effects of vegetative resistance in river flows. Several such relations exist that relate the average flow velocity to the hydraulic roughness and vegetation geometry (Strickler, 1923; Nikuradse, 1933; Keulegan, 1938).

Existing studies are driven by the needs in the construction and restoration of wetlands (Kadlec, 1990) and flood plains (Mertens, 1989). Measures to restore streams to their natural standing, e.g. planting of trees and shrubs along the flood plains, can significantly alter their hydraulic properties and the quantifying of these effects is a very important task. Most researches in this regard have been devoted to the overall hydraulic characteristics of vegetated channels (Pasche, 1984; Rickert, 1986; Klaassen and Van der Zwaard, 1974; Wu et al., 1999; Armanini and Rigetti, 1998; Rouve, 1987).

Notable works within the realm of velocity distribution in vegetated channels are those of Tsujimoto and Kitamura (1992 a, b; 1997; 1998) and Kutija and Hong (1996). Yet, most of these works fail to present vertical velocity distribution and its comparison with measurements in the case of stiff vegetation. At the same time experimental investigations clearly show that the velocity distribution inside vegetation is nearly uniform, especially further from the channel bed (Rowinski et al., 1998).

Knowledge of the vertical velocity distributions may be helpful in the estimates of shear velocities and, consequently, bed shear stresses. These parameters are key factors in estimating the bed load transport and the related scour, deposition, entrainment and bed changes in rivers; shear velocities also being traditionally used for normalization of basic parameters of turbulence. Recognition of the hydraulic processes of flows in vegetated channels is still at a preliminary stage. Vegetation influences the resistance and velocity distribution of the watercourse considerably, creates additional drag exerted by plants, causes a violent transverse mixing due to great differences in velocities in vegetated and non-vegetated regions, and also affects the turbulence intensity and diffusion (Nepf, 1999; Nikora, 2000; Rowinski et al., 1998).

Experimental campaigns have been set up to measure flow characteristics in natural vegetated streams (Green, 2006). Experiments in laboratory flumes have been carried out, with some recent studies (Armanini et al., 2005; Jaarvelaa, 2002; Stephan and Gutknecht, 2002). Also, detailed numerical simulations of flow through vegetation were performed (Shimizu and Tsujimoto, 1994; Erduran and Kutija, 2003; Neary, 2003; Choi and Kang, 2004). As a result of such studies, several relationships have been proposed that describe flow velocity distribution and resistance as caused by vegetation. Most of these relationships are empirical (Ree and Crow, 1977; Kouwen and Fathi-Moghadam, 2000), some of them benefiting from stronger theoretical foundations (Petryk and Bosmajian, 1975; Stone and Shen, 2002). Also, there are methods that are based on modified logarithmic velocity distribution (Kouwen and Unny, 1973; Stephan and Gutknecht, 2002). Klopfstra et al. (1997) combined methodologies by treating flow over vegetation in a two-layer approach, where flow in each of the two layers is described separately. They combined a modified logarithmic velocity distribution (in the surface layer) with a newly derived velocity profile that is present in the vegetation zone.

Among the existing velocity distribution and resistance relationships in vegetated channels, the empirical ones have the advantage of being simple in form. On the other hand, empirical relationships suffer from the drawback that their applicability is limited to the range of conditions for which they were derived. Theoretical descriptions...
An Analytical Model for Velocity Distribution

Main Channel Flow

Logarithmic Zone

Transition Zone

Uniform Zone

Figure 1. Velocity distribution and forces on element in transition zone in inflexible submerged vegetative channels

are often complex. Besides, they may require poorly understood closure parameters and they sometimes pose practical difficulties when gathering required input data.

All these processes have been studied extensively, but many questions remain unanswered that constitute the basis for an important debate.

A useful approach for studying flow problems in vegetated areas is to attempt to construct a relative analytical model for the description of the vertical velocity distribution, which could replace the traditional empirical ones. In the present study, an idealized case is considered. The velocity profiles discussed occur on vegetated channels. The vegetations are assumed to be cylindrically shaped and uniformly distributed in a square pattern at equal spacing in both longitudinal and transverse directions which is typical of many restored streams (Klaassen and Van der Zwaard, 1974; Rouve, 1987). It is expected that the equidistant arrangement of the vegetation should not affect the results essentially; rather that the density being a crucial factor.

In the present study, an analytical model is developed for assessing velocity distribution along the transition zone, and for the transition zone thickness.

Theoretical Background

In water flow through inflexible vegetated channels, the transition zone, being located in between the logarithmic and uniform zones, has not been so thoroughly investigated. As all the previous researches have focused on empirical relationships with limitations in application, this study was planned to be carried out in a way as to introduce the velocity distribution of the transition zone in submerged inflexible vegetated channels with a view on (unique) mathematical relationships. Figure 1 shows velocity distribution in vegetation zone (uniform zone), transition zone as well as in the main channel (logarithmic zone).

In inflexible submerged vegetative channels

To establish the mathematical description for the transition zone, the force equilibrium equation for an element in the transition zone and for a unit of channel width is written as:

$$\tau_{\text{r1}} - \tau_i - F_D + W \sin \alpha = 0$$  \hspace{1cm} (1)
where $\tau_{i+1}$ and $\tau_{i}$ are water shear stresses, $F_D$ is the drag force per unit area and $W \sin \alpha$ the gravity force component per unit area. According to the Figure 1, $\Delta x$, $\Delta y$ and $\Delta z$ are the dimensions of element in the longitudinal, width wise, and vertical directions respectively. $F_D$ can be expressed as (Klopstra et al., 1997):

$$F_D = \frac{1}{2} C_D \rho N D u^2 \Delta y$$  \hspace{1cm} (2)

where $F_D$ is the drag force per unit area, $C_D$ is the coefficient of drag, $\rho$ is water density, and $N$ the number of stems per unit area, $D$ is the stem diameter and $u$ the flow velocity at the point of the element. In Figure 1, $y$ is the water depth measured with respect to top of vegetation.

$W \sin \alpha$ can be written as:

$$W \sin \alpha = \frac{\gamma \Delta x \Delta y S}{\Delta x} = \gamma S \Delta y$$  \hspace{1cm} (3)

where $\gamma$ is the specific weight and $S$ the channel longitudinal slope.

Substituting (3) and (2) in (1) gives:

$$\tau_{i+1} - \tau_{i} - \frac{1}{2} C_D \rho N D u^2 \Delta y + \gamma S \Delta y = 0$$  \hspace{1cm} (4)

then:

$$\Delta \tau - \frac{1}{2} C_D \rho N D u^2 \Delta y + \gamma S \Delta y = 0$$  \hspace{1cm} (5)

Writing Equation (5) in the differential form gives:

$$\frac{d \tau}{dy} - \frac{1}{2} C_D \rho N D u^2 y + \gamma S = 0$$  \hspace{1cm} (6)

**Mixing Length Concept**

There are several possible ways to describe the shear stress term (Nezu and Nakagawa, 1993), but most of them are relatively complex. Here, a simple hypothesis connecting the eddy viscosity to the flow conditions through a mixing length concept is employed. The fluid shear stress can be expressed as:

$$\tau = \eta \frac{du}{dy}$$  \hspace{1cm} (7)

where $\tau$ is the fluid shear stress, $\eta$ eddy viscosity and $\frac{du}{dy}$ is the velocity gradient.

Using the Prandtl Mixing Length, eddy viscosity can be described as follows:

$$\eta = \rho l^2 \frac{du}{dy}$$  \hspace{1cm} (8)

where $l$ is the Prandtl Mixing Length. Combining (7) and (8) will introduce:

$$\tau = \rho l^2 \left( \frac{du}{dy} \right)^2$$  \hspace{1cm} (9)

Among the numerous expressions to evaluate mixing length, the most popular was proposed by Nikuradse (Nezu and Nakagawa, 1993) and further applied in open channels. Nikuradse’s expression can be conveniently simplified to a linear form as:

$$l = \kappa d$$  \hspace{1cm} (10)

where $\kappa$ is Von Karman constant and $d$ the transition zone thickness. This form has been widely used in open channels, and under uniform turbulent flow it has led to a well known expression for mean velocity logarithmic vertical profile. However, obstacles to flow (such as vegetation) generate turbulent eddies and may limit the use of the open channel methods. Further from the channel bed, generation of such eddies by vegetation can predominate. Thus linear dependence for the mixing length is valid only at some distance from the bed and is constant beyond that threshold value. This distance supposedly depends on the density of vegetation.

Substituting (10) in (9), the shear stress can be written as:

$$\tau = \rho \kappa^2 d^2 \left( \frac{du}{dy} \right)^2$$  \hspace{1cm} (11)

Differentiating Equation (11) with respect to $y$ will introduce:

$$\frac{d \tau}{dy} = \rho \kappa^2 d^2 \frac{d}{dy} \left( \frac{du}{dy} \right)^2$$  \hspace{1cm} (12)
By substituting (12) in (6) the differential equation to be solved will be obtained as:

$$\kappa^2 d^2 \left( \frac{du}{dy} \right)^2 + gS - \frac{1}{2} C_D N u^2 = 0 \quad (13)$$

Solution of the above equation will give the velocity distribution in transition zone.

**Boundary Conditions**

Shear stress at the top of vegetation can be calculated through the following equations:

$$\tau_{y=0} = \rho \kappa^2 d \left( \frac{du}{dy} \right)^2 \quad (14)$$

$$\tau_{y=0} = \rho u_*^2 \quad (15)$$

where $u_*$ is the shear velocity. The origin of $y$-axis is shown in Figure 1.

Combining (14) and (15) gives:

$$\frac{du}{dy} \bigg|_{y=0} = \frac{u_*}{\kappa d} \quad (16)$$

At $y = 0$ velocity can be found from the logarithmic velocity distribution in the main channel flow, thus:

$$u \bigg|_{y=0} = u_0 \quad (17)$$

where $u_0$ is the velocity at the top of vegetation. In the case where the height of vegetation is more than the momentum penetration depth from the top flow velocity (at the end of transition zone ($y = d$)) will generally be uniform and as:

$$\frac{du}{dy} \bigg|_{y=d} = 0 \quad (18)$$

The uniform velocity inside the vegetation can be calculated by applying the force equilibrium equation as follows:

$$- \frac{1}{2} C_D \rho N D u_t^2 \Delta y + gS \Delta y = 0 \quad (19)$$

$$u_t = \sqrt{\frac{2gS}{C_D N D}} \quad (20)$$

therefore:

$$u \bigg|_{y=d} = u_1 = \sqrt{\frac{2gS}{C_D N D}} \quad (21)$$

where $u_1$ is the uniform velocity at the bottom of vegetation zone.

**Analytical Solution**

Analytical solution of Equation (13) can be achieved by introducing intermediate variable, $t$, as follows:

$$\frac{du}{dy} = t \quad (22)$$

$$\frac{dt}{dy} = 2t \quad (23)$$

$$\frac{2\kappa^2 d^2 t^2}{du} dt = \frac{1}{2} C_D N D u^2 - gS \quad (24)$$

Integrating the above equation gives:

$$\frac{2}{3} \kappa^2 d^2 t^3 = \frac{1}{6} C_D N D u^3 - gS u + A \quad (26)$$

where $A$ is the integration constant which can be calculated using Equations (18), (19) and (22) according to the following:

$$A = gS u_t - \frac{1}{6} C_D N D u_1^3 \quad (27)$$

Substituting (22) and (27) in (26) will result in:

$$\frac{2}{3} \kappa^2 d^2 \left( \frac{du}{dy} \right)^3 = \frac{1}{6} C_D N D (u^3 - u_t^3) - gS (u - u_t) \quad (28)$$

By substituting (16) and (17) in (28), the transition zone thickness will be calculated as:

$$d = \frac{\frac{2}{3} u_t}{\frac{1}{6} C_D N D (u_t^3 - u_0^3) - gS (u_t - u_0)} \quad (29)$$

To establish the velocity distribution relationship for transition zone, Equation (28) can be written in the following form:

$$\frac{dy}{du} = \left( \frac{2}{3} C_D N D \left( u^3 - u_t^3 \right) - gS (u - u_t) \right)^{1/3} \quad (30)$$

Substituting (29) in (30) gives:
\[ \frac{dy}{du} = \left( \frac{1}{6} C_D ND(u^3 - u_1^3) - gS(u - u_1) \right)^{1/3} \left( \frac{1}{6} C_D ND(u_0^3 - u_1^3) - gS(u_0 - u_1) \right)^{2/3} \]

Introducing:

\[ A_0 = \frac{1}{6} C_D ND \]
\[ B_0 = \frac{1}{6} C_D ND \]
\[ C_0 = -gS \]
\[ D_0 = gS u_1 - \frac{1}{6} C_D ND u_1^3 \]

While replacing (32), (33), (34) and (35) in (31) and integrating, the final equation will be obtained as:

\[ y = -\int_{u_1}^{u_0} \frac{A_0}{(B_0 u^4 + C_0 u + D_0)^{1/3}} du \] (36)

Equation (36) does not have a closed form of solution and therefore, to be solved numerical solutions applied (e.g. trapezoidal or Simpson rule) should be applied.

**Model Calibration and Verification**

The performance of the analytical model is assessed in three successive steps of:

1. Conducting experiments,
2. Calibrating the model parameters using a part of the experiments,
3. Verifying the model precision using another part of the experiments.

**Experiments**

Experiments have been made in a rectangular laboratory flume with a length of 15, width of 0.3, and a height of 0.45 m, equipped with an adjustable slope. Instead of channel inflexible vegetation, solid stems made of stiff plastic were employed in an identical staggered layout (Figure 2). Stems were used in two heights of 5 and 10 cm and placed at 3 cm spaces longitudinally as well as transversely.

A number of 30 experiments were conducted using different discharges and depths and in each of which the vertical velocity distribution measured through a micro-propeller.

**Figure 2.** The staggered layout of stems.

**Model Calibration**

Sixty percent of experiments were used for model calibration. Values of \( C_D \) and \( \kappa \) have been calculated through optimization. The selection of the experiments for calibration and verification stages was done through a random selection. The objective function was defined as:

\[ \sum (u_{cal} - u_{obs})^2 \]

which was minimized using the conjugate search method. The values of \( C_D \) and \( \kappa \) were obtained as equal to 1.93 and 0.21 respectively. \( C_D \) is affected by arrangement and diameter of vegetation. According to Klopstra et al. the value of \( C_D \) in this condition is between 1.0 and 2.0 (Klopstra et al., 1997). Von-Karman coefficient (\( \kappa \)) is a parameter which changes according to the turbulence structure. It expresses the turbulence scale, and for normal water condition in open channels, it is equal to 0.41. For different conditions of turbulent scales inside the vegetation, it varies from 0.1 to 0.25 (Kouwen and Unny, 1973).

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\[ \text{(31)} \]

\[ \text{(32)} \]

\[ \text{(33)} \]

\[ \text{(34)} \]

\[ \text{(35)} \]

\[ \text{(36)} \]
An Analytical Model for Velocity Distribution

Figure 3. Observed and calculated velocity distribution for a discharge of: 3.67 lit sec\(^{-1}\), Water depth = 92 mm, Slope = 1:60 and Vegetation height = 5 cm.

**Model Verification**

The rest of the experiments were used for model verification. In the verification stage the velocity distribution and transition zone thickness have been calculated and compared with the observed values.

**RESULTS AND DISCUSSION**

Figures 3 to 7 show the results obtained from the model in the calibration and verification stages. It should be noted that according to Figure 1, the direction of y-axis for transition zone is downward and subsequently the values of y for transitional velocity profile negative.

Figure 3. Observed and calculated velocity distribution for a discharge of: 3.67 lit sec\(^{-1}\), Water depth = 92 mm, Slope = 1:60 and Vegetation height = 5 cm.

Figure 4. Observed and calculated velocity distribution for a discharge of: 5.72 lit sec\(^{-1}\), Water depth = 156.2 mm, Slope = 1:500 and Vegetation height = 10 cm.

Figure 5. Observed and calculated thicknesses of transition zone in all experiments.

Figure 6. Observed and calculated velocities in the verification stage.

Figure 7. Observed and calculated velocities in all experiments.

**Sensitivity Analysis**

To check the model sensitivity to optimized parameters (\(C_D\) and \(\kappa\)), sensitivity analysis of the model for the transition zone thickness \(d\) was conducted, the results being as follows:

**Table 1. Model sensitivity analysis.**

<table>
<thead>
<tr>
<th>Variations</th>
<th>(C_D)</th>
<th>(\kappa)</th>
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<tbody>
<tr>
<td>+2%</td>
<td>-2%</td>
<td>+5%</td>
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<tr>
<td>+5%</td>
<td>-5%</td>
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Figure 4. Observed and calculated velocity distribution for a discharge of: 5.72 lit sec\(^{-1}\), Water depth = 156.2 mm, Slope = 1:500 and Vegetation height = 10 cm.
Figure 5. Observed and calculated thicknesses of transition zone in all experiments.

Figure 6. Observed and calculated velocities in the verification stage.

Figure 7. Observed and calculated velocities in all experiments.
As observed from Table 1 model sensitivity with respect to $C_D$ and $\kappa$ is in an acceptable range.

**CONCLUSIONS**

In the present study, submerged flow over inflexible vegetation elements is described by an analytical model where flow velocity distribution is attributed to three separate zones of: the bottom uniform distribution zone, upper logarithmic distribution zone and intermediate transition zone. There are extensive literature items about bottom uniform and logarithmic zones, but less investigations are available concerning transition zones. Existing literature about transition zones are mostly either empirical or numerical.

In the present study the force equilibrium equation is used to analytically construct the expression for transition zone velocity distribution. The simple drag force relationship and the Prandtl Mixing Length concept were employed as components of the force equilibrium equation. It was shown that the obtained differential equation can be analytically solved. Comparing the model results with the observed data proved that the model is able to accurately predict the transition zone thickness and its velocity distribution pattern. The advantages of the proposed model are: use of the well known basic fluid mechanics principles, straightforward derivation, an exact calculation of the transition zone thickness, not accurately possible through other models, and finally a good precision of the model.

The employed vegetation simulations in the experiments were some cylindrical stiff stems of uniform diameter which when compared with natural vegetation of non-uniform diameter and/or sparse densities may cause variations in model’s accuracy and precision.

**Nomenclature**

$\tau_{i+1}$ Shear stress on upper side of the element;

$\tau_i$ Shear stress on bottom of the element;

$\Delta x$ Water element size in the longitudinal direction;

$\Delta y$ Water element size in the channel width direction;

$\Delta z$ Water element size in the vertical direction;

$y$ Water depth;

$F_D$ Drag force per unit of area;

$W \sin \alpha$ Weight component in the direction of the movement;

$C_D$ Coefficient of drag;

$\rho$ Water density;

$N$ Number of stems per unit of area;

$D$ Stem diameter;

$u$ Velocity along element;

$\gamma$ Water specific weight;

$S$ Longitudinal slope;

$\tau$ Shear stress between fluid layers;

$\eta$ Eddy viscosity;

$\frac{du}{dy}$ Velocity gradient;

$l$ Prandtl Mixing Length;

$\kappa$ Von Karman constant;

$d$ Transition zone thickness;

$u_s$ Shear velocity;

$u_0$ Velocity at the top of vegetation;

$u_1$ Uniform velocity at bottom of vegetation;

$t$ Intermediate variable;

$A$ Integration constant;

$A_0, B_0, C_0, D_0$ Substitutive variables;

$u_{cal}$ Calculated velocity;

$u_{obs}$ Observed velocity.
REFERENCES


مدل تحلیلی ناحیه انقلابی در توزیع سرعت در آبراههای با پوشش گیاهی انعطاف‌نابین مسیری

ج. م. و. سامانی و م. مظاهری

چکیده

در مجاری با پوشش گیاهی انعطاف‌نابینی و بستری مسیری، تخمین مقدار سرعت و دبی حائز اهمیت می‌باشد. با توجه به در بالاترین آب، توزیع سرعت در این گونه آبراهه‌ها به سه بخش قابل تقسیم می‌باشد: بخش یک‌واخت ناحیه که به کف آبراهه تندیک است و دارای سرعت یک‌واخت می‌باشد، بخش فوقانی که بدون پوشش گیاهی می‌باشد و از قانون توزیع لگاریتمی سرعت تبیعی می‌کند و بخش میانی (انتقالی) که تحت تأثیر مونتوم قسمت فوقانی قرار می‌گیرد. در این مداری روش‌های موجود برای برآورد مقدار دیوار و سرعت بر اساس سرعت متوسط می‌باشد. در این مقاله روش تحلیلی جهت محاسبه عمق و توزیع سرعت در ناحیه انقلابی مذکور ارائه گردیده است. نتایج بدست آمده حاکی از تطبیق خوب نتایج این مدل با نتایج داده‌های آزمایش‌گاهی است.