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Analytical Solution to One-dimensional Advection-diffusion Equation with Several Point Sources through Arbitrary Time-dependent Emission Rate Patterns

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ABSTRACT

Advection-diffusion equation and its related analytical solutions have gained wide applications in different areas. Compared with numerical solutions, the analytical solutions benefit from some advantages. As such, many analytical solutions have been presented for the advection-diffusion equation. The difference between these solutions is mainly in the type of boundary conditions, e.g. time patterns of the sources. Almost all the existing analytical solutions to this equation involve simple boundary conditions. Most practical problems, however, involve complex boundary conditions where it is very difficult and sometimes impossible to find the corresponding analytical solutions. In this research, first, an analytical solution of advection-diffusion equation was initially derived for a point source with a linear pulse time pattern involving constant-parameters condition (constant velocity and diffusion coefficient). Hence, using the superposition principle, the derived solution can be extended for an arbitrary time pattern involving several point sources. The given analytical solution was verified using four hypothetical test problems for a stream. Three of these test problems have analytical solutions given by previous researchers while the last one involves a complicated case of several point sources, which can only be numerically solved. The results show that the proposed analytical solution can provide an accurate estimation of the concentration; hence it is suitable for other such applications, as verifying the transport codes. Moreover, it can be applied in applications that involve optimization process where estimation of the solution in a finite number of points (e.g. as an objective function) is required. The limitations of the proposed solution are that it is valid only for constant-parameters condition, and is not computationally efficient for problems involving either a high temporal or a high spatial resolution.

Keywords: Advection-diffusion equation, Analytical solution, Laplace transformation, Point source, Solute transport.

INTRODUCTION

Advection-Diffusion Equation (ADE) describes the transport of solute under the combined effects of advection and diffusion. This equation is a parabolic partial differential one derived as based upon the conservation of mass and Fick's first law. ADE benefits from wide applications in such different disciplines as environmental engineering, mechanical engineering, heat transfer, soil science, petroleum engineering, chemical engineering and as well in biology. ADE interprets the spreading of scalar or non-scalar quantities under specified initial and boundary conditions. This equation can be solved either analytically or numerically. There are several schemes for solving ADE numerically which do not fall within the scope of this paper.

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Analytical solutions are as useful tools in many areas (van Genuchten, 1981; Jury *et al*., 1983; Leij *et al*., 1991; Quezada *et al*., 2004). They can be applied for providing preliminary or approximate analyses of alternative pollution scenarios, conducting sensitivity analyses to investigate the effects of various parameters or processes on contaminant transport, extrapolation over large times and distances where numerical solutions may be impractical, serving as screening models or benchmark solutions for more complex transport processes that cannot be accurately solved, and for verifying more comprehensive numerical solutions of the governing transport equations (Guerrero *et al*., 2009).

Formerly, the analytical solutions of ADE were obtained by reducing the original ADE into a diffusion equation by omitting the advective terms (Kumar *et al*., 2009); this being carried out either by applying moving coordination (Ogata and Banks, 1961; Harleman and Rumer, 1963; Bear, 1972; Guvanasen and Volker, 1983; Aral and Liao, 1996; Marshal *et al*., 1996); or by introducing another dependent variable (Banks and Ali, 1964; Ogata, 1970; Lai and Jurinak, 1971; Marino, 1974; Al-Niami and Rushton, 1977; Kumar *et al*., 2009). In more recent works, analytical solutions of ADE have been obtained using such integral transform techniques as either Laplace or Fourier transforms (Govindaraju and Bhabani, 2007). In fact these transformations can act as powerful tools in solving differential equations. Van Genuchten and Alves (1982) provided a comprehensive manual of ADE analytical solutions for different initial and boundary conditions, utilizing Laplace transform technique. Other researchers have widely applied this technique in solving ADE analytically (Smedt, 2007; Kazezyılmaz-Alhan, 2008; Kumar *et al*., 2009; Kumar *et al*., 2010). In addition to either Laplace or Fourier transform methods, Hankel transform method, Aris moment method, perturbation approach, methods using Green's function, superposition method and mathematical substitutions have also been applied to provide analytical solutions for ADE (Courant and Hilbert, 1953; Morse and Feshbach, 1953; Carslaw and Jaeger, 1959; Sneddon, 1972; Ozisik, 1980; Zwillinger, 1998; Leij and van Genuchten, 2000; Polyanin, 2002; Kumar *et al*., 2009). Guerrero *et al*. (2009) have presented some algorithms to derive formal exact solutions of ADE using change-of-variable and integral transform techniques. Cotta (1993) developed Deneral Integral Transform Technique (GITT). This method allows derivation of analytical or semi-analytical solution of the more general form of ADE's. In the GITT, the solution is written in terms of eigenfunction series expansions. There exist some researches regarding analytical solutions of ADE using GITT (Chongxuan *et al*., 2000; Barros *et al*., 2006; Guerrero and Skaggs, 2010).

Most of the mathematically closed form analytical solutions of ADE are limited to simple initial and boundary conditions. The more complexity of boundary conditions leads to complicated mathematical formulae that may involve numerically solvable integrals, special functions or infinite summations, and in many cases an analytical solution cannot be found. In this paper, an analytical solution of one-dimensional constant-parameter ADE is initially derived for linear pulse time-dependent boundary conditions. Using the superposition principle, the solution can be extended to an analytical one for any desired time-dependent boundary conditions. Applying this method, the mathematically closed form formula can be derived for any number of point sources of the desired time-dependent emission rate patterns.

Analytical Solution of 1D ADE for Linear Pulse Time-dependent Boundary Condition

Here, the analytical solution of the one-dimensional ADE for linear pulse time-dependent boundary condition is derived using Laplace transform. The ADE is considered with constant parameters where the decay is also taken into account. The ADE with these properties is as follows:

$$\frac{\partial C}{\partial t} = -U\frac{\partial C}{\partial x} + D\frac{\partial^2 C}{\partial x^2} - kC, \quad 0 < x < \infty, \quad 0 < t < \infty \tag{1}$$

In which, C is the solute concentration, U the constant flow velocity, D is the constant diffusion coefficient, k the coefficient of first-

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order reaction, while t and x representing the variables of time and space respectively. Equation (1) is solved with respect to the following initial and boundary conditions:

$$C(x,0) = C_0 \tag{2}$$

$$C(0,t) = (at+b)\left[u(t-t_1) - u(t-t_2)\right] \tag{3a}$$

$$\left.\frac{\partial C}{\partial x}\right|_{x\to\infty} = 0 \tag{3b}$$

Where, C_0 is the initial concentration, a and b are the parameters of the linear pulse boundary condition at $x=0$, t_1 and t_2 are the beginning and ending times of the source activation, respectively, while $u(t-t_i)$ being the shifted Heaviside function, defined to be 0 for $t<t_i$ and 1 for $t \geq t_i$. Equation (3a) is depicted in Figure 1.

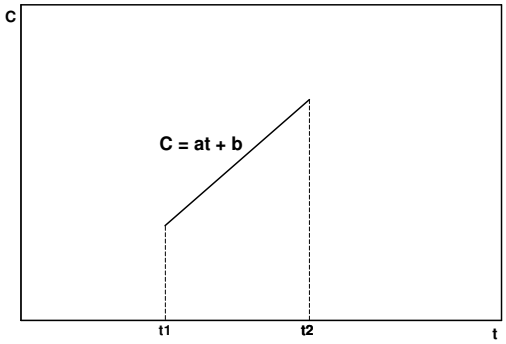

Figure 1. Depiction of the input boundary condition.

Applying the Laplace transform to Equation (1) and its boundary conditions yields the corresponding problem in the Laplace domain; that is:

$$L\left(\frac{\partial C}{\partial t}\right) = L\left(-U\frac{\partial C}{\partial x} + D\frac{\partial^2 C}{\partial x^2} - kC\right) \tag{4}$$

$$L\left(C(0,t)\right) = L\left((at+b)\left[u(t-t_1)-u(t-t_2)\right]\right) \tag{5a}$$

$$L\left(\left.\frac{\partial C}{\partial x}\right|_{x\to\infty}\right) = L(0) \tag{5b}$$

Where, L is the Laplace transform operator. Using the forward Laplace transform formulas (Abramowitz and Stegun, 1970), Equations (4) and (5) can be written as an ordinary differential equation in the Laplace domain:

$$s\bar{C} - C(x,0) = -U\frac{d\bar{C}}{dx} + D\frac{d^2\bar{C}}{dx^2} - k\bar{C} \tag{6}$$

$$\bar{C}(0,s) = (at_1+b)\frac{\exp(-t_1 s)}{s} + a\frac{\exp(-t_1 s)}{s^2} - (at_2+b)\frac{\exp(-t_2 s)}{s} - a\frac{\exp(-t_2 s)}{s^2} \tag{7a}$$

$$\left.\frac{d\bar{C}}{dx}\right|_{x\to\infty} = 0 \tag{7b}$$

In which $\bar{C} = \bar{C}(x,s)$ is the corresponding dependent variable of Equation (1) in the Laplace domain and s representing the Laplace transform variable. Equation (6) is a linear inhomogeneous ordinary differential equation that is of the following general solution:

$$\bar{C}(x,s) = c_1 \exp\left(\frac{Ux}{2D} + \sqrt{\frac{U^2x^2}{4D^2} + \frac{kx^2}{D} + \frac{sx^2}{D}}\right) + c_2 \exp\left(\frac{Ux}{2D} - \sqrt{\frac{U^2x^2}{4D^2} + \frac{kx^2}{D} + \frac{sx^2}{D}}\right) + \frac{C_0}{s+k} \tag{8}$$

Where, c_1 and c_2 are arbitrary constants, that can be specified using Equations (7a) and (7b). Using these equations, c_1 and c_2 would be as follows:

$$c_1 = 0 \tag{9a}$$

$$c_2 = (at_1+b)\frac{\exp(-t_1 s)}{s} + a\frac{\exp(-t_1 s)}{s^2} - (at_2+b)\frac{\exp(-t_2 s)}{s} - a\frac{\exp(-t_2 s)}{s^2} - \frac{C_0}{s+k} \tag{9b}$$

Assuming $\alpha = U^2x^2/(4D^2) + kx^2/D$, $\beta = x^2/D$ and $\gamma = Ux/(2D)$, and replacing the Equations (9a) and (9b) in Equation (8), this equation is simplified to:

$$\bar{C}(x,s) = \left[(at_1+b)\frac{\exp(-t_1 s)}{s} + a\frac{\exp(-t_1 s)}{s^2} - (at_2+b)\frac{\exp(-t_2 s)}{s} - a\frac{\exp(-t_2 s)}{s^2} - \frac{C_0}{s+k}\right] \times \exp(\gamma)\exp\left(-\sqrt{\alpha+\beta s}\right) + \frac{C_0}{s+k} \tag{10}$$

Now, the analytical solution to Equation (1) with respect to (2) and (3) can be achieved by returning Equation (10) back from the Laplace

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domain to the time domain. This can be done by obtaining the inverse Laplace transform of Equation (10). Using the linearity property of inverse Laplace transform operator, the inverse Laplace transform of Equation (10) can be written as:

$$
\begin{aligned}
&L^{-1}\left(\bar{C}(x,s)\right)=(at_1+b)\exp(\gamma)\times \\
&L^{-1}\left(\exp(-t_1 s)\frac{\exp\left(-\sqrt{\alpha+\beta s}\right)}{s}\right)+ \\
&a\exp(\gamma)\times \\
&L^{-1}\left(\exp(-t_1 s)\frac{\exp\left(-\sqrt{\alpha+\beta s}\right)}{s^2}\right)- \\
&(at_2+b)\exp(\gamma)\times \\
&L^{-1}\left(\exp(-t_2 s)\frac{\exp\left(-\sqrt{\alpha+\beta s}\right)}{s}\right)- \\
&a\exp(\gamma)\times \\
&L^{-1}\left(\exp(-t_2 s)\frac{\exp\left(-\sqrt{\alpha+\beta s}\right)}{s^2}\right)- \\
&C_0\exp(\gamma)\times \\
&L^{-1}\left(\frac{\exp\left(-\sqrt{\alpha+\beta s}\right)}{s+k}\right)+C_0 L^{-1}\left(\frac{1}{s+k}\right)
\end{aligned}
\tag{11}
$$

Where, L^{-1} is the inverse Laplace transform operator. The inverse Laplace transforms appearing in Equation (11) could be solved according to the inverse Laplace transform formulas (Abramowitz and Stegun, 1970). Therefore, the final solution for different values of t would be as the following:

$$
\begin{aligned}
&C(x,t)=-\frac{C_0}{2}\exp(-kt)\operatorname{erfc}\left(\frac{\beta-2\gamma t}{2\sqrt{\beta t}}\right) \\
&-\frac{C_0}{2}\exp(2\gamma-kt)\operatorname{erfc}\left(\frac{\beta+2\gamma t}{2\sqrt{\beta t}}\right)+ \\
&C_0\exp(-kt), \qquad 0\le t<t_1
\end{aligned}
\tag{12a}
$$

$$
\begin{aligned}
&C(x,t)=\exp\left(\gamma-\sqrt{\alpha}\right)\left[a\left(\frac{t-t_1}{2}-\frac{\beta}{4\sqrt{\alpha}}\right)\right. \\
&\left.+\frac{at_1+b}{2}\right]\operatorname{erfc}\left(\frac{\beta-2(t-t_1)\sqrt{\alpha}}{2\sqrt{\beta(t-t_1)}}\right)+ \\
&\exp\left(\gamma+\sqrt{\alpha}\right)\left[a\left(\frac{t-t_1}{2}+\frac{\beta}{4\sqrt{\alpha}}\right)+\frac{at_1+b}{2}\right] \\
&\times\operatorname{erfc}\left(\frac{\beta+2(t-t_1)\sqrt{\alpha}}{2\sqrt{\beta(t-t_1)}}\right)-\frac{C_0}{2}\times \\
&\left[\exp(-kt)\operatorname{erfc}\left(\frac{\beta-2\gamma t}{2\sqrt{\beta t}}\right)+\exp(2\gamma-kt)\times\right. \\
&\left.\operatorname{erfc}\left(\frac{\beta+2\gamma t}{2\sqrt{\beta t}}\right)\right]+C_0\exp(-kt)
\end{aligned}
\tag{12b}
$$

$$
\begin{aligned}
&C(x,t)=\exp\left(\gamma-\sqrt{\alpha}\right)\left[a\left(\frac{t-t_1}{2}-\frac{\beta}{4\sqrt{\alpha}}\right)\right. \\
&\left.+\frac{at_1+b}{2}\right]\operatorname{erfc}\left(\frac{\beta-2(t-t_1)\sqrt{\alpha}}{2\sqrt{\beta(t-t_1)}}\right)+ \\
&\exp\left(\gamma+\sqrt{\alpha}\right)\left[a\left(\frac{t-t_1}{2}+\frac{\beta}{4\sqrt{\alpha}}\right)+\right. \\
&\left.\frac{at_1+b}{2}\right]\operatorname{erfc}\left(\frac{\beta+2(t-t_1)\sqrt{\alpha}}{2\sqrt{\beta(t-t_1)}}\right)- \\
&\exp\left(\gamma-\sqrt{\alpha}\right)\left[a\left(\frac{t-t_2}{2}-\frac{\beta}{4\sqrt{\alpha}}\right)+\right. \\
&\left.\frac{at_2+b}{2}\right]\operatorname{erfc}\left(\frac{\beta-2(t-t_2)\sqrt{\alpha}}{2\sqrt{\beta(t-t_2)}}\right)- \\
&\exp\left(\gamma+\sqrt{\alpha}\right)\left[a\left(\frac{t-t_2}{2}+\frac{\beta}{4\sqrt{\alpha}}\right)+\right. \\
&\left.\frac{at_2+b}{2}\right]\operatorname{erfc}\left(\frac{\beta+2(t-t_2)\sqrt{\alpha}}{2\sqrt{\beta(t-t_2)}}\right)- \\
&\frac{C_0}{2}\left[\exp(-kt)\operatorname{erfc}\left(\frac{\beta-2\gamma t}{2\sqrt{\beta t}}\right)+\right. \\
&\left.\exp(2\gamma-kt)\operatorname{erfc}\left(\frac{\beta+2\gamma t}{2\sqrt{\beta t}}\right)\right]+ \\
&C_0\exp(-kt)
\end{aligned}
\tag{12c}
$$

Where, erfc(.) is the complementary error function. Equations (12a), (12b) and (12c) are established for a point source at $x=0$. For a general point source at location $x=x_s$, variable

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x must be replaced by $x - x_s$. In this case Equations (12a), (12b) and (12c) can be interpreted as a unified equation using the shifted Heaviside function. Simplifying and performing the necessary algebraic manipulations would yield to:

$$\begin{aligned} &C(x,t,x_s,t_1,t_2) = A(x-x_s,t,t_1,-1) + \\ &A(x-x_s,t,t_1,1) - A(x-x_s,t,t_2,-1) - \\ &A(x-x_s,t,t_2,1) - B(x-x_s,t,-1) - \\ &B(x-x_s,t,1) + 2B(0,t,0) \end{aligned} \tag{13}$$

In which, x_s is the source location and $A(.)$ and $B(.)$ are functions defined according to the following:

$$\begin{aligned} &A(x,t,t_i,m) = u(x)u(t-t_i)\times \\ &\exp\left(\frac{Ux}{2D} + m\sqrt{\frac{U^2x^2}{4D^2} + \frac{kx^2}{D}}\right)\times \\ &\left[a\left(\frac{t-t_i}{2} + \frac{mx}{2\sqrt{U^2+4Dk}}\right) + \frac{at_i+b}{2}\right]\times \\ &\operatorname{erfc}\left(\frac{x+m(t-t_i)\sqrt{U^2+4Dk}}{2\sqrt{D|t-t_i|}}\right) \end{aligned} \tag{14}$$

$$\begin{aligned} &B(x,t,m) = \frac{C_0}{2}u(x)\times \\ &\exp\left(-kt + (1+m)\frac{Ux}{2D}\right)\operatorname{erfc}\left(\frac{x+mUt}{2\sqrt{Dt}}\right) \end{aligned} \tag{15}$$

Equation (13) is the analytical solution for a linear pulse point source at $x = x_s$. Note that Equation (13) yields to 0 for $x < x_s$. Equations (13), (14) and (15) in this form are suitable for computer implementation. In Equation (13), the terms expressed, using function $A(x,t,t_i,m)$ consider the effects of advection, diffusion and decay of pollutant released from a point source while the terms expressed, using function $B(x,t,m)$ consider the effects of advection, diffusion and decay of initially existing concentration at $t = 0$.

Extending the Solution for a More Realistic Point Source Emission Time Pattern

In this section the derived analytical solution is extended for any such desired piecewise linear time pattern as the one depicted in Figure 2.

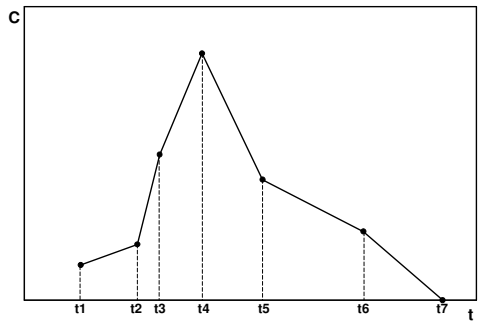

Figure 2. A typical piecewise linear time pattern.

This can be done according to the superposition principle, meaning that a source with a piecewise linear time pattern may be considered as several linear units and the effects of these units being additive. In this case the solution can be written as:

$$\begin{aligned} &C(x,t,x_s) = \sum_{i=1}^{nl}[A(x-x_s,t,t_i,-1) + \\ &A(x-x_s,t,t_i,1) - A(x-x_s,t,t_{i+1},-1) - \\ &A(x-x_s,t,t_{i+1},1)] + 2B(0,t,0) - \\ &B(x-x_s,t,-1) - B(x-x_s,t,1) \end{aligned} \tag{16}$$

Where, $C(x,t,x_s)$ is the concentration value at (x,t) due to a point source at location x_s, nl is the number of linear units in the source time pattern while t_i and t_{i+1} are the beginning and ending times of the ith linear unit respectively, and functions $A(.)$ and $B(.)$ having been described before. It is clear that in computing the values of function $A(.)$ for each i in Equation (16), the corresponding parameters (a_i and b_i) must be applied. Note that in Equation (16) the terms expressed by function $B(.)$ are out of the summation as the effects of initial conditions must be considered as only once. For a point source with emission time pattern that does not follow a piecewise linear pattern, it is possible to fit an approximate piecewise linear function and then use Equation (16). The number of linear segments in the fitting function could be increased for an accurate capturing of the original pattern.

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Solution for Several Point Sources

Using the superposition principle for several point sources, it is possible to extend Equation (16) for several point sources as follows:

$$C(x,t)=\sum_{j=1}^{ns}\sum_{i=1}^{nl}\Big[A(x-x_{sj},t,t_{j,i},-1)+ A(x-x_{sj},t,t_{j,i},1)-A(x-x_{sj},t,t_{j,i+1},-1)- A(x-x_{sj},t,t_{j,i+1},1)-A(x-x_{sj},t,t_{j,i+1},1)\Big]+ 2B(0,t,0)-B(x,t,-1)-B(x,t,1) \quad (17)$$

Where, $C(x,t)$ is the concentration at (x,t) due to all point sources, ns is the number of point sources and j the subscript that encounters the jth source. Once again the terms in $B(.)$ function are out of the summation in Equation (17) and the variable x in this function must be considered with respect to the minimum available value in x-coordinate.

In fact using the property of linearity of Equation (1) and superposition principle in time and space, the solution can be well constructed. The final solution for several point sources with different arbitrary time patterns can be expressed as a simple function that is straight forward for computer implementation. The only possible difficulty in computer implementation these formulas is computing the complementary error function ($\mathrm{erfc}(.)$). However, most of computer packages and libraries in relevant disciplines are capable of computing the function.

RESULTS AND DISCUSSION

In this section, the obtained analytical solution is verified. Verification of analytical solutions developed by other researchers has been carried out using hypothetical examples and numerical methods (Smedt, 2007; Kazezyılmaz-Alhan, 2008; Williams and Tomasko, 2008; Kumar *et al.*, 2010). In this paper verification is conducted using analytical solutions proposed in the literature in addition to some more complicated hypothetical examples. In the latter case, the presented analytical solution will be compared with the numerical solution. Here, the solutions are given for a stream flow by velocity, U, of 0.7 m s^{-1}, width, B, of 20 m, depth, h, of 1.5 m and longitudinal slope, S_0, of 0.0005. The dispersion coefficient, D, is calculated from Fischer's formula (1979) ($D=0.011U^2B^2/(h\sqrt{9.81hS_0})$) equal to 16.8 m^2 s^{-1}.

Test Problem I: Constant Pulse Point Source

Figure 3 shows a constant pulse point source *vs.* time with a duration of 1 hour starting from t_1= *1* to t_2= *2* hr and intensity of W_s= 5 kg s^{-1}. The values of a and b for this source are 0 and 0.24 kg m^{-3}, respectively. Note that, the relationship between point source mass discharge, W_s, and its concentration, C_s, is $C_s=W_s/Q$ where $Q=UBh$ is the stream volumetric discharge.

Other parameters are: x_s= *1 km*, $C_0=0$, $k=0$ and the whole stream length is considered as equal to 4 km. Figure 4 shows the results of the proposed analytical solution and a classic analytical solution (van Genuchten and Alves, 1982) for the whole stream length within different times.

As depicted in Figure 4, the results of the proposed analytical solution and the one given by van Genuchten and Alves (1982) are exactly the same. In fact the classic solution can be derived from Equation (13) by letting $a=0$ and b equal to the source intensity.

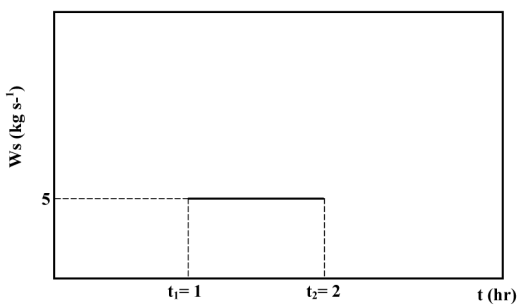

Figure 3. Constant pulse point source emission time pattern (test problem I).

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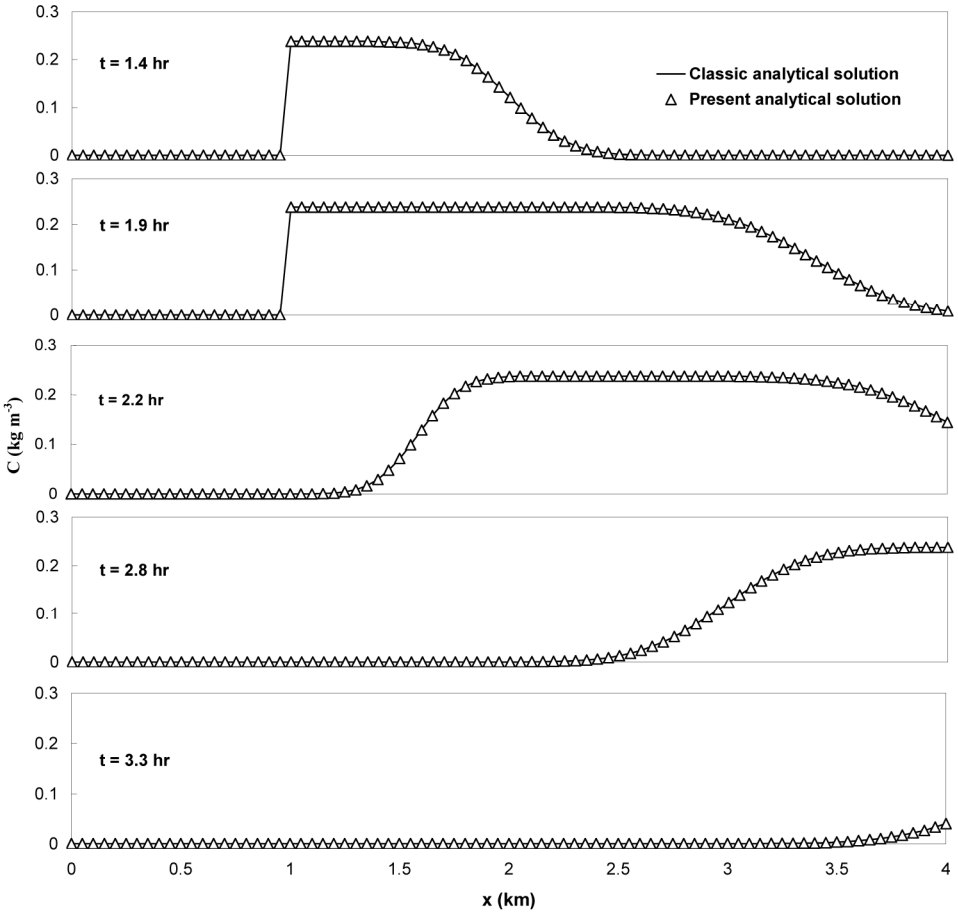

Figure 4. Results of the present analytical and classic solutions for test problem I.

Test Problem II: Instantaneous Spill

The solution of instantaneous spill of a mass in a stream has been given by Fischer *et al.* (1979). The point source emission time pattern in this condition could be considered as a constant pulse with a very short duration. It is clear that the area under mass discharge time pattern curve must be equal to the total mass spilled (Figure 5). Figure 6 shows the results of the proposed analytical and classic solutions for a mass spill problem. The magnitude of 1000 kg of a conservative material is spilled into the stream at $t= 0$. Other conditions are similar to those in the test problem I. The duration for this problem (Δt) is considered as equal to 0.1 second. As depicted in Figure 6, there exists a reasonable agreement between the two solutions. A small difference occurred, which is caused by the fact that the classic analytical solution to this problem has been derived for some ideal conditions. In other words the mass is spilled instantaneously. It is evident that this assumption is somehow unrealistic and in the real situations the duration must be duly considered.

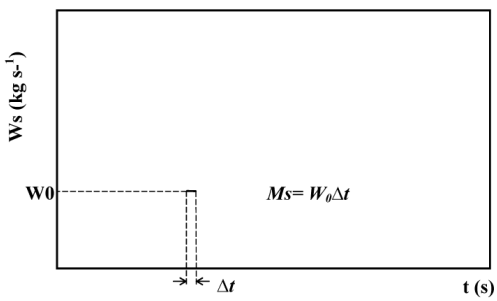

Figure 5. Instantaneous spill of a mass as a very short constant pulse (test problem II).

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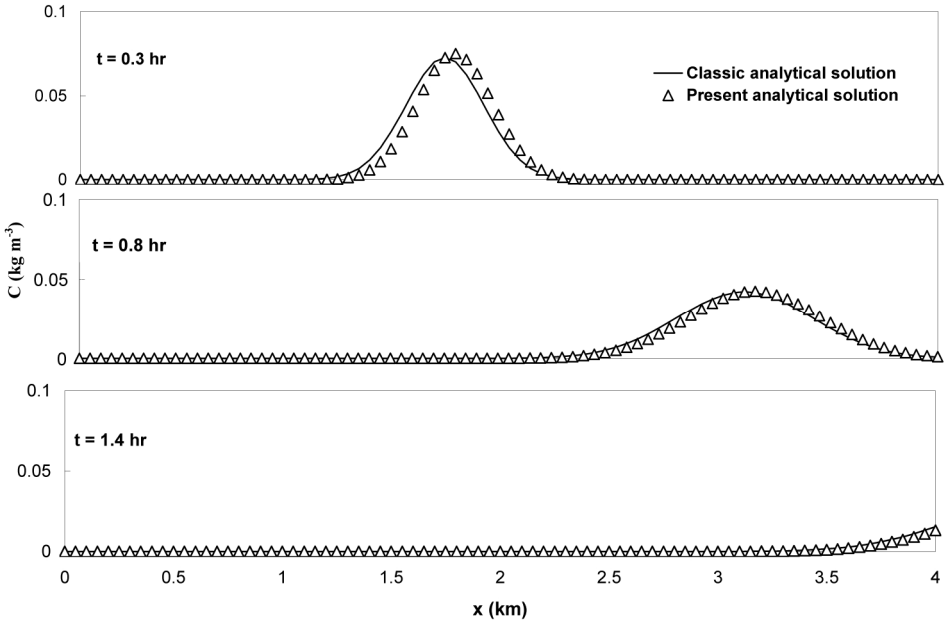

Figure 6. Results of the present analytical and classic solutions for test problem II.

Test Problem III: Decaying Point Source

Decaying point sources are decayed exponentially with respect to time. The relationship for this kind of sources can be considered as $W = W_0 \exp(-\lambda t)$, in which W_0 is the initial intensity and λ the decay coefficient. An analytical solution for decaying point source has been given by van Genuchten and Alves (1982). To model decaying point sources, using the proposed analytical solution, a piecewise linear function must be fitted to the exponential pattern. Figures 7a-b show the exponential pattern with $W_0 = 5\,kg/s$ and $\lambda = 1/3600$ and the fitted piecewise linear function by 3 and 6 linear units, respectively. Other conditions are the same as in previous test problems. The length of the stream under study is assumed to be 8 km, and it is assumed that the source has started at $t = 0$. The results of the present analytical solution and the classic analytical solution are shown in Figures 8 and 9. As observed from Figure 8, there is a small difference between the two solutions. This is due to the difference between the real source pattern and the fitted function. The results show that by increasing the number of

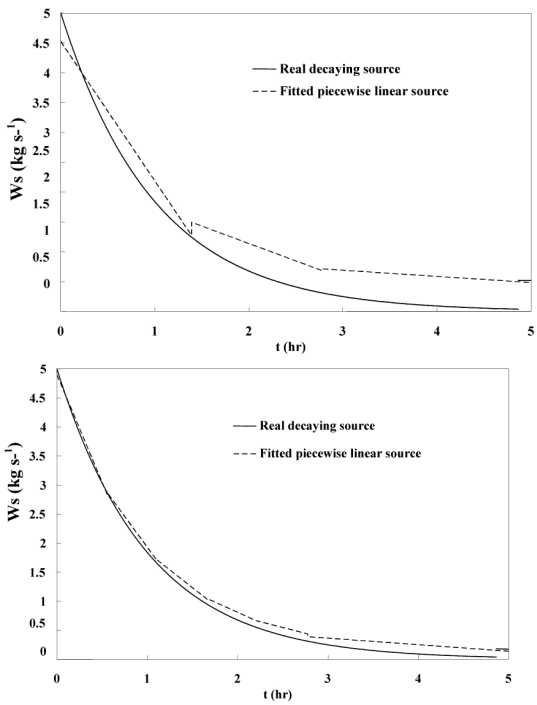

Figure 7. Decaying point source and fitted 3-unit (a) fitted 6-unit (b) piecewise linear function (test problem III).

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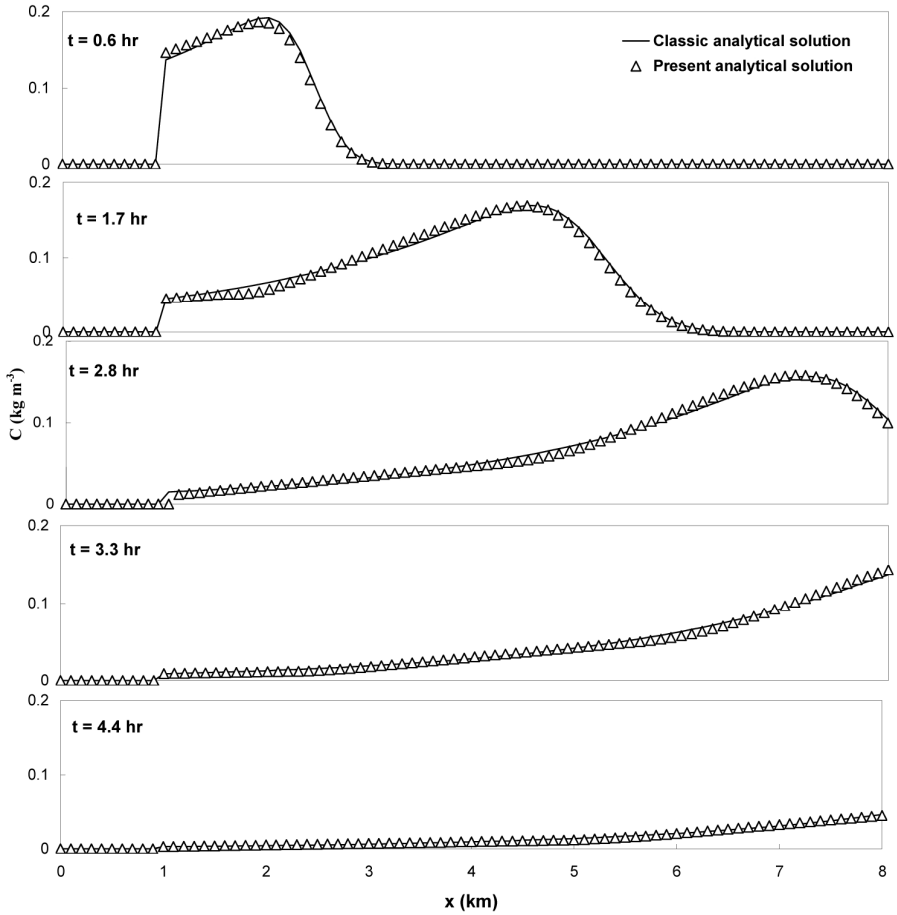

Figure 8. Results of the present analytical and classic solutions for test problem III (3-unit piecewise linear function).

linear units in the fitted function and more precisely by capturing of the real pattern, the discrepancy between the two solutions could be reduced.

Test Problem IV: Several Point Sources with Irregular Patterns

In this section, the derived analytical solution is applied for a complicated case. The case includes 4 point sources with different patterns. In this test problem, since no analytical solution has been previously developed for such cases, the analytical solution given in this paper is compared against a numerical solution. The numerical method used, is an implicit-scheme finite difference by central discritization estimation for spatial derivatives. Figure 10 shows the emission time patterns for 4 point sources used in this test problem. It was tried to select the patterns in a way for covering more variability. The sources are located at 2, 4, 10 and 15 km and a stream length of 30 km is considered. All other parameters are as in the previous test problems. Time step and spatial grid spacings for numerical solution are considered as equal to 10 seconds and 20 meters respectively. Figure 11 shows the results of the proposed analytical solution and the numerical solution for this test problem. As depicted in Figure 11, the proposed analytical solution is of the potential to be used in the accurate computation of the concentration in time and space. A small difference appearing is due to the inevitable error in the numerical solution. The obtained analytical solution could include initial concentration and first-order reaction decaying in the computations. Figure 12 shows

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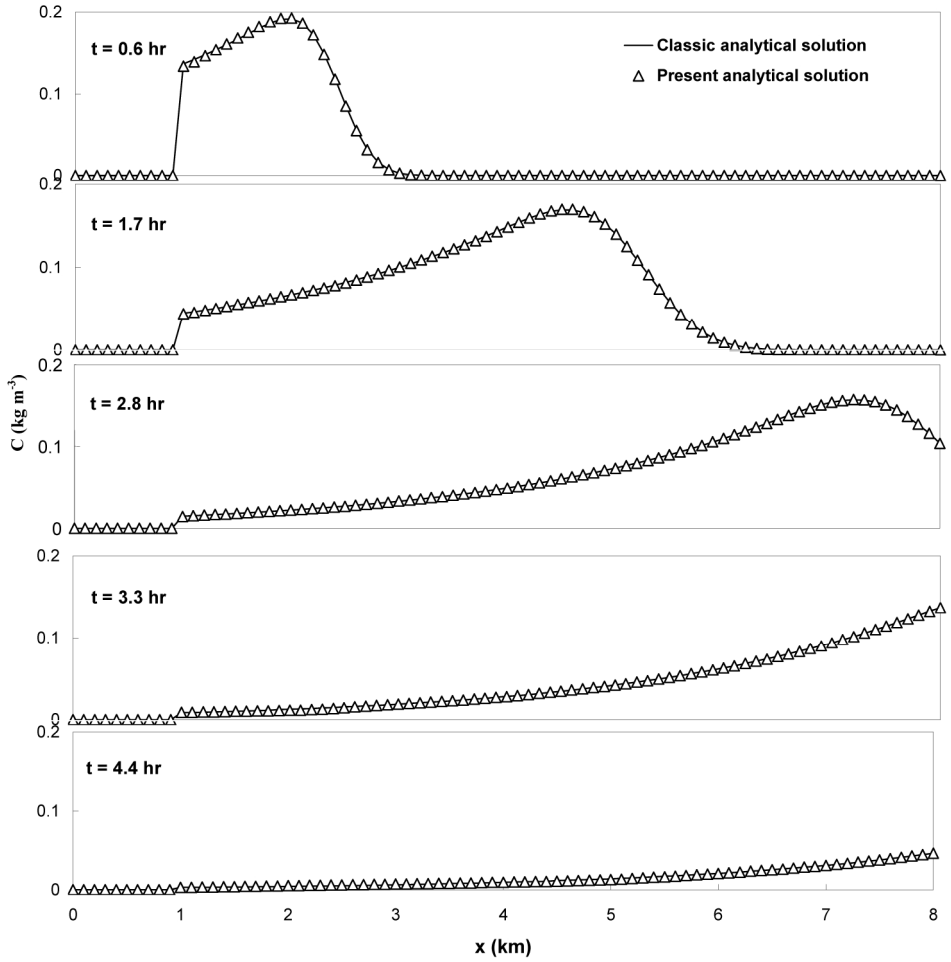

Figure 9. Results of the present analytical and classic solutions for test problem III (6-unit piecewise linear function).

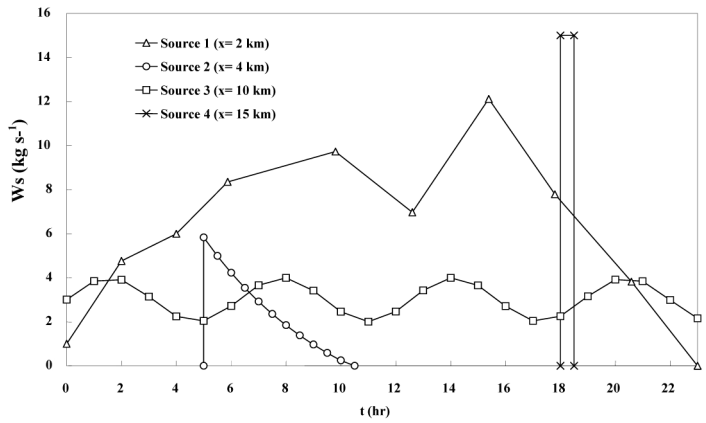

Figure 10. Patterns used for point sources in test problem IV.

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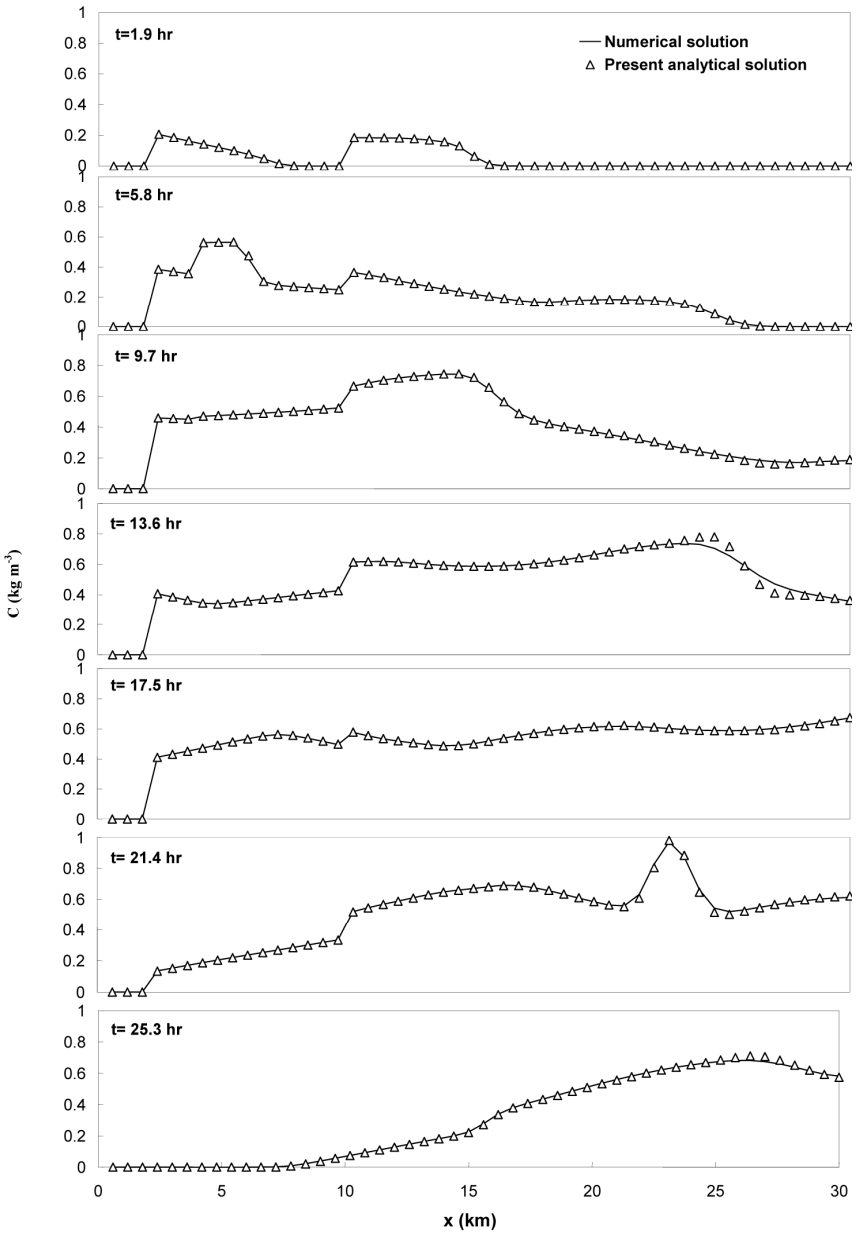

Figure 11. Results of the proposed analytical solution and numerical solution for test problem IV (C_0= 0 and k= 0).

the results of the test problem IV for C_0= *0.3 kg m*$^{-3}$ and k= *0.00001 s*$^{-1}$.

CONCLUSIONS

An analytical solution of ADE for linear pulse point source was obtained using Laplace transform technique. Based on this solution the superposition principle was employed to extend the derived solution for several point sources through arbitrary time patterns. The proposed method is straight forward for computer implementation. The method was applied for different test problems and the results demonstrate that it is highly accurate. The presented solution is of various applications; it can be used instead of

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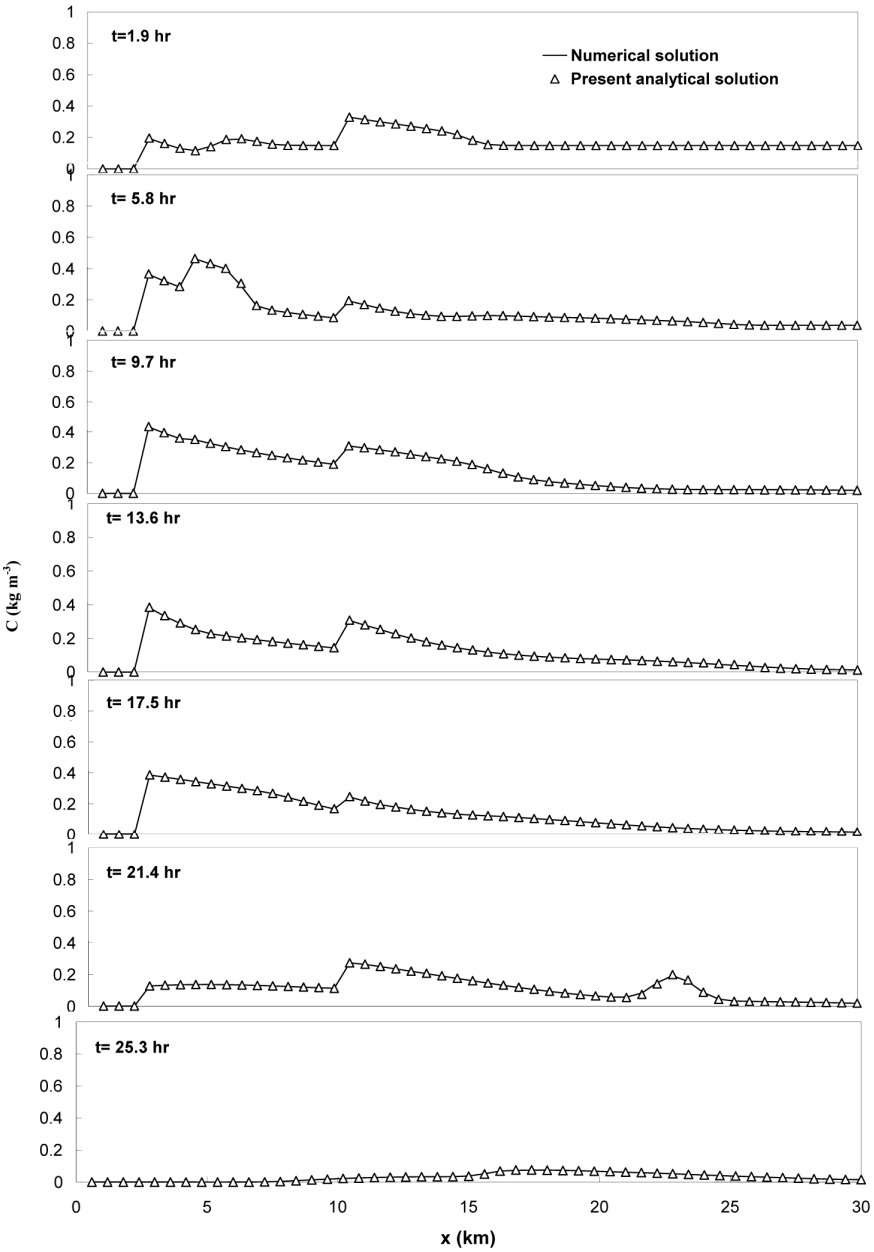

Figure 12. Results of the proposed analytical solution and numerical solution for test problem IV (C_0= 0.3 kg m^{-3} and k= 0.00001 s^{-1}).

numerical models for constant-parameter conditions and also for verifying complicated transport codes. While straight forward to implement, the proposed method is exact and fast. Furthermore, unlike numerical models which are typically faced with such difficulties as stability, accuracy, computational costs, etc., the proposed method does not suffer from such issues. The analytical solution given in this paper is suitable for cases where the ADE should be solved over large temporal or spatial intervals or in cases where the solution needs to be estimated only in some specific points. This occurs when the analytical solution is applied as the objective function in optimization processes. In spite of advantages, the proposed method suffers from some limitations. The most important limitation is

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that the presented solution is only valid when the parameters (U, D) are constants. The other limitation is that, for high temporal or spatial grid resolution, the computational costs may exceed those of the numerical methods.

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حل تحلیلی معادله یک بعدی جابه‌جایی-پخشیدگی درحالت وجود چند منبع آلاینده نقطه‌ای با الگوهای زمانی دلخواه

م. مظاهری، ج. م. و. سامانی، و ح. م. و. سامانی

چکیده

معادله جابه‌جایی-پخشیدگی و حل‌های تحلیلی آن در زمینه‌های متفاوتی دارای کاربرد می‌باشند. در مقایسه با حل‌های عددی، حل‌های تحلیلی دارای مزیت‌هایی هستند. تا به حال حل‌های تحلیلی متفاوتی برای معادله جابه‌جایی-پخشیدگی ارائه شده است. تفاوت عمده در این حل‌ها عمدتاً به نوع (الگوی زمانی) شرایط مرزی در نظر گرفته شده برای آنها باز می‌گردد. تاکنون تقریباً در تمامی حل‌های تحلیلی ارائه شده برای این معادله، شرایط مرزی به صورت بسیار ساده در نظر گرفته شده است. این در صورتی است که برای حالت‌های کاربردی معمولاً شرایط مرزی پیچیده بوده و به دست آوردن حل تحلیلی در

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این حالت بسیار مشکل و در اکثر موارد غیرممکن می‌گردد. در این تحقیق در ابتدا حل تحلیلی معادله جابه‌جایی-پخشیدگی با ضرایب ثابت برای یک منبع نقطه‌ای با الگوی زمانی پالس خطی استخراج می-گردد. سپس با استفاده از اصل جمع آثار حل استخراج شده برای حالت الگوی زمانی دلخواه و برای چند منبع آلاینده تعمیم داده می‌شود. در مرحله بعد حل تحلیلی استخراج شده برای چهار حالت فرضی در رودخانه به کار گرفته شده است. سه حالت از حالات فرضی در نظر گرفته شده دارای حل‌های تحلیلی بوده که توسط محققین قبلی ارائه شده‌اند. حالت چهارم، حالت پیچیده‌ای است که در آن چند منبع آلاینده با الگوهای زمانی پیچیده وجود دارد و قبل از ارائه حل تحلیلی موجود در این تحقیق، فقط با استفاده از روش‌های عددی قابل حل می‌باشد. نتایج به کار بردن حل تحلیلی ارائه شده برای حالات مذکور حاکی از عملکرد درست و دقت مناسب آن بوده و در نتیجه می‌توان از آن در کاربردهای مختلف و از جمله صحت‌سنجی مدل‌های پیچیده عددی انتقال و یا به عنوان تابع هدف برای مقاصد تخمین پارامترهای انتقال بهره برد. از محدودیت‌های حل تحلیلی ارائه شده می‌توان به حاکم بودن آن فقط در حالت ضرایب (سرعت، ضریب پخشیدگی و ضریب واکنش) ثابت اشاره نمود. همچنین در حالاتی که شبکه مکانی یا زمانی برای تخمین آن بزرگ است، ممکن است روش ارائه شده از نظر محاسباتی نسبت به روش‌های عددی به صرفه نباشد.

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